2017 John O'Bryan Mathematics Competition
5-Person Team Test

1. Complete each of the following (5 points each part):

   a. Assume $x \neq 1$. Find all solutions to the equation: $\left(x^2+x+1\right)\left(x^6+x^3+1\right)=\frac{10}{x-1}$

   b. Let $f(x)=x^2+ax-9$ and $g(x)=x^2+bx+c$ $-$ where $a, b,$ and c are constants. Suppose that $f(x)$ has roots r and s while $g(x)$ has roots $-r$ and $-s$. Find the roots of $f(x)+g(x)$

2. Let $T(x)$ be a non-zero polynomial. Under what conditions on T do we have each of the following properties?

   a. $T(x(\ ))=1$

   b. $T(x(\ ))=x$

   c. $T(x(\ ))=T(x)$

   d. $T(x(\ ))=\left(T(x)\right)^2$

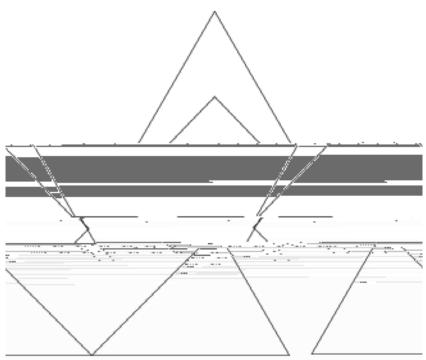
3.

4. The numbers from 1 to 1000 are written, in order, in a large circle. Starting at the number 1, every r^{th} number (1, 1+r, 1+2r, etc.) is crossed out. This is continued until a number is reached that has already been crossed out.

   a. If r = 15, what is the total number of cross-outs?
   b. In general, what is the total number of cross-outs?

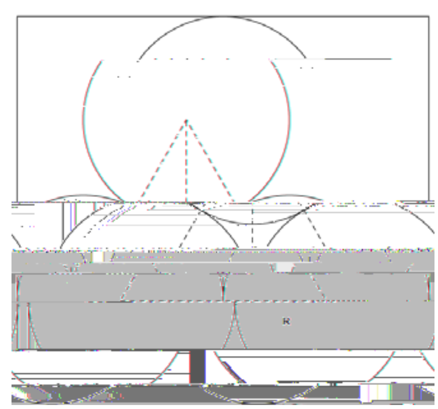
5. An equilateral triangle with sides of length 2 will have a square placed inside of it.

   a. If one side of the square sits on one side of the triangle, find the area of the largest such square.
   b. If the square is oriented such that one diagonal of the square is collinear with one of the vertices of the triangle (as shown below, left), find the area of the largest such square.
   c. Finally, if the triangle is as shown below (right), with $0 \le \theta \le 30$ degrees, find a formula for the area of the largest such square at angle θ.



6. In the figure shown, the circles are tangent to one another and to the sides of the rectangle. Each of the circles has radius R.

   a. What is the area of the entire rectangle?
   b. What is the area of the region trapped between the three circles?



Solutions to team exam :

Note to coaches : I have inserted some comments after a few of the problems o may find these sef in preparing yo r teams in the f t re I noticed that a ot of teams chose methods that were more diffic t than needed It might be sef for teams to spend time oo ing at each problem and g essing at possib e best strategies as a team before dividing p problems to be so ved by individ a s

1 a. Assuming $x \neq 1$, find the solutions to $x^2 + x + 1 \quad x^6 + x^3 + 1 = \frac{10}{x-1}$

Solution :Multiply (x-1) times ($x^2 + x + 1$) to get ^{3} 1. Now multiply this times ($x^6 + x^3 + 1$) to get

^{9} 1=10 and so x = $\sqrt[9]{11}$

Note: any teams did this by m tip ying the eft side o t getting a ar ge mess and then simp ifying down Notice how reco gni ing the difference of c bes ma es this prob em a ot easier saving time and e iminating many possib e errors

b. If $f(x) = x^2 + bx - 9$ and $g(x) = x^2 + dx - e$, and if f has two roots r and s, while g has roots $-r$ and $-s$, find the roots of $f(x) + g(x) = 0$.

Solution:

f(x)=(x-r)(x-s) =(x-r)(x-s)= ^{2} *r s* rs and so rs =-9. Likewise g(x) = (x+r)(x+s))= ^{2} *r s* rs, so *f g* 2 ^{2} 2 rs 2 ^{2}-9) =0 gives x= 3.

Note: ot of teams g essed that b and d were 9 and finished the so tion from there o need to show this since there are a ot of possibi ities for these two n m bers

2. Let P (x) be a non-zero polynomial. Under what conditions on P do we have each of the following properties?

a. P(P(x)) = 1

b. . P(P(x)) = x

c. . P(P(x)) = P(x)

d. . P(P(x)) = (^{2}

Solution :

a. A polynomial has form P(x) = a_n ^{n} $a_{n\ 1}$ $^{n\ 1}$ ... a_1 $a_{0,}$

b. Again, looking at the highest power, we must have that $a_n\ a_n^{\ n\ n}\ \ a_n^{n\ 1\ \ n^2}$=x so n=1, and then a_1^2=1, so a_1 1 or a_1 1.

Case when a_1 1. Then P(x) =x+c, for some number c. Then P(P(x)) = (x+c)+c = x+2c, so c=0.

Case when a_1 1. then P(x)=-x+c and P(P(x)) = -(-x+c) +c = x. So P(x)=-x+c works for any x.

c. If the highest power of P(x) is n, ie P(x) = $a_n\ ^{n}\ \ a_{n\ 1}\ ^{n\ 1}$ ... $a_1\ \ a_{0,}$ then P(P(x)) has highest degree that looks like $a_n\ a_n^{\ n\ n}\ \ a_n^{n\ 1\ \ n^2}$. We want this to be equal to $a_n\ ^{n}$. This says that n^2 = n, so that n=0 or n=1. The n=0 case is the constant function P(x)=1.

In the case when n=1, we get that a (a x +b) + b =(a x + b) or a^2 ab b ax b. For a^2 a, we have a 0 back to the constant function , or a=1.Then from the constant terms we have 2b=b, so b=0. So the only polynomials possible are P(x) =x and P(x) = C, for any real number C.

d. In this case from the highest powers we have $a_n\ a_n^{\ n\ n}\ \ a_n^{n\ 1\ \ n^2}$=
Subscript a, n ^n ^2 $a_n^{2\ \ 2n}$. This is only possible if 2 n n^2, so n 0 constant function) or n=2.

If P(x)=c, then P(P(x)) =c while ^{2} c^2. So the only constant function is c=1 (or zero).

If n=2, then Subscript a, 2 ^3 Subscript a, 2 ^{2}, so a_2 1. If P(x) = ^2 ax b, then ^{2} ^{4}+2a ^{3} 2 b a^2 ^{2}+2 a b x + b^2.
Also P(P(x)) = ^{4} 2 a ^{3}+(a + a^2 b^2) ^{2}+ a^2 b + a b + b^2. Setting the coefficients equal we get the system of equations b(a+1)=0, 2 ab a^2, and 2 b a b^2. From the first equation b=0, or a=-2. If b=0, then from the second equation a=0 as well, so ^{2}. If a=-2, then the second equation gives b=-1, but this solution does not satisfy the last equation.

So P(x)=1 or ^{2}

Note: On this problem many teams guessed some solutions t in mathematics we also want to know if we have all the solutions o should learn to ask this question and try to give an argument about why you have all the solutions.

3. You are given two sizes of ceramic tiles. There are 1 x 1 tiles, in white and red colors, and 1 x 3 tiles, in blue, green and orange colors. You can make patterns by stringing tiles together. For example you can make a 1 x 6 tile of red, orange, red, white tiles. Of course you could also tile a 1 x 6 tile using blue followed by green. You are also allowed to be boring and tile the 1 x 6 using all red tiles, if you wish. We also count using the same colors, but in a different order, as a new tiling.

How many different tilings are there of a 1 x n tiles, for n = 1,2,3,4,5, and 6?

Solution:

Let T(n) = number of tilings of a 1 x n tiling.

For n= 1, we can use one of the the white or red tiles. So 2 possible. T(1)=2.
For n=2 we have two choices for the leftmost tiles and 2 for the next tile, so T(2) = 2 x 2 = 4.
For n=3, we can make the leftmost tiling a 1 x 1 tiling (2 choices), then we are left with a 1x2 tile to file, so 2 x T(2) = 8. Or we can put down a 1 x 3 tiling (3 choices) and be done. So T(3) = 2 x T(2) + 3 = 11

Note: ot of teams contin ed in this way, wor in g to 45 and e ow is a way to wor o t a of these at once sin g a f nction echnica y htis is ca ed sin g rec rsion and is a common y idea to simplify a proced re in both mathematics and comp ter pro gammin g

Now for any larger tiling we can proceed as for n=3: If we start with a 1x1 tiling we have a 1 x (n-1) area left to tile, so 2 x T(n-1) ways to tile. If we start with a 1 x 3 we have, in the same way, 3 x T(n-3) ways to tile. So T(n)=2T(n-1)+3T9n-3). Using this we get:
T(4)=2 x 11+3x2= 28 , and likewise T(5)=68 and T(6)=169.

4. The numbers from 1 to 1000 are written, in order, in a large circle. Starting at 1, every r-th number (1,r+1, 2r+1, etc) is crossed out. This is continued until a number is reached that has already been crossed out.

a. If r=15 what is the total number of numbers that have been crossed out?

b. In general, what is the total number of cross-outs?

Solution :

a. The crossed out numbers are 1,16, 31, ... 991=1+ 15 x 66
and second time around 6, 21, 36, ., 996 =6 + 15x66
and the third time around 11,26,41, 986 = 11+15 x 65
the next number would be 986+15-1000 = 1, our first repeat.

There were 67+67+66 = 200 numbers crossed out all together.

b. Suppose they match after n steps and on the nth step we are the same as an earlier mth step. Then we are asking when the numbers (1+nr)-(1+mr) are divisible by 1000. This is equivalent to (n-m) r being divisible by 1000.

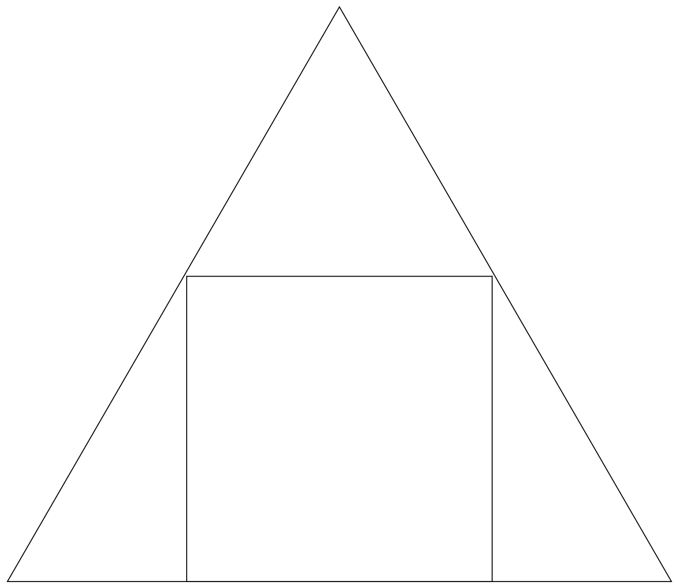
i) If r and 1000 have no common divisors, then n-m will be divisible by 1000, a so must be 1000. So, in this case, all the numbers will be crossed out.

ii) Suppose GCD(1000, r) is the greatest common divisor of r and 1000. Since (n-m) r = (n-m) GCD(1000, r) p =1000 k, then we have that n-m = 1000/GCD(1000,r). So the number crossed out is 1000/GCD(1000,r).

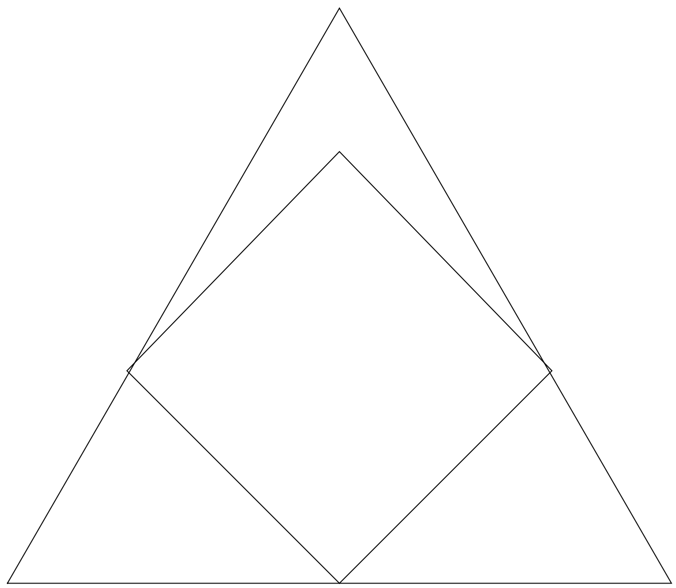
Note that in the previous case r= 15 and GCD(1000,15)=5. So the number crossed out is 1000/5 = 200.

5. The largest square possible is placed in an equilateral triangle of side-length 2.

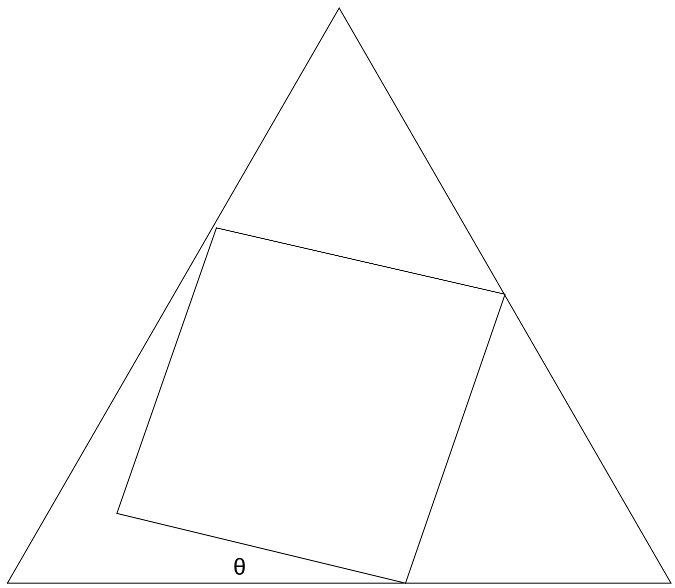
a. If one side of the square sits on one side of the triangle, find the area of the largest such square.



b. If the triangle is oriented as shown, then find the area of the largest such square.

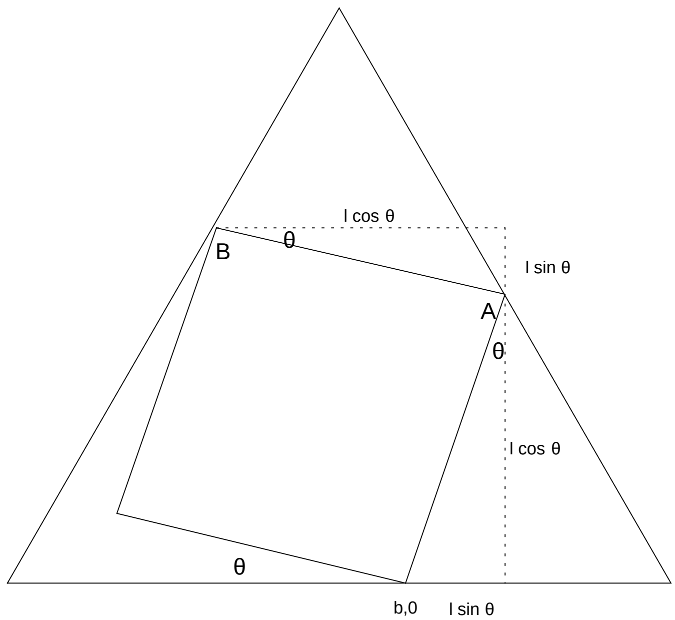


c. Finally, if the triangle is as shown, with $0 \leq \theta \leq 30$, find a formula for the area of the largest such triangle with square at the angle θ.



Solutions :

a.



Then the point labeled A must have coordinates (b+l sin θ , l cos θ) and the point labeled B is (b+l sin θ - l cos θ , l cos θ+ l sin θ).

Since A is on the line y= $-\sqrt{3}\ (x - 1)$, we can plug in the coordinates (b+l sin θ , l cos θ) and simplify to l(cos θ + $\sqrt{3}$ sin θ) = $\sqrt{3}$ - $\sqrt{3}$ b.

Likewise, since B is on the line y $\sqrt{3}$ $\sqrt{3}$, we can get that l ((1+ $\sqrt{3}\big)\cos\theta\ \ \big(1\ \ \sqrt{3}\big)\sin\theta\ \ \sqrt{3} + \sqrt{3}\ b$

Adding these two equations results in l ($\cos\theta\left(2\ \ \sqrt{3}\right)\ \ \sin\theta\Big)\ \ 2\sqrt{3}$ or, l= $\frac{2\sqrt{3}}{\left(2\ \sqrt{3}\right)\cos\theta\ \ \sin\theta)}$. Then the area is 2 Sqrt 3 2 Sqrt 3 cos θ sin θ ^{2}.

Checking against the previous computations: (Using Mathematica)

$$f\ \theta\ \ :\ \frac{2\sqrt{3}}{2\ \ \sqrt{3}\ \ \mathrm{Cos}\ \theta\ \ \ \mathrm{Sin}\ \theta}$$

Checking part a (θ =0)

$$f\ 0\ \hat{}2\ \ \ \frac{12}{7\ \ 4\sqrt{3}}$$

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True
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Checking part b (θ = 15 degrees = π/12 radians, where θ f 12.